

POLYNOMIAL CONFIGURATIONS IN SETS OF POSITIVE UPPER DENSITY OVER LOCAL FIELDS

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ABSTRACT. Let \mathbb{K} be the field of real or p -adic numbers, and $F(x) = (f_1(x), \dots, f_m(x))$ be such that $1, f_1, \dots, f_m$ are linearly independent polynomials with coefficients in \mathbb{K} . Employing ideas of Bachoc, DeCorte, Oliveira and Vallentin in combination with estimating certain oscillatory integrals with polynomial phase we will show that the independence ratio of the Cayley graph of \mathbb{K}^m with respect to the portion of the graph of F defined by $a \leq \log |s| \leq T$ is at most $O(1/(T - a))$. From here, we conclude that if $I \subseteq \mathbb{K}^m$ has positive upper density, then the difference set $I - I$ contains vectors of the form $F(s)$ for an unbounded set of values $s \in \mathbb{K}$. We deduce that the Borel chromatic number of the Cayley graph of \mathbb{K}^m with respect to the set $\{\pm F(s) : s \in \mathbb{K}\}$ is infinite, while the clique number of this Cayley graph is finite. Moreover, the infiniteness of the Borel chromatic number does not hold necessarily if f_1, \dots, f_m are merely real-analytic functions.

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1. INTRODUCTION

Let A be a non-compact abelian topological group, equipped with an invariant distance \mathbf{d} . Fix a symmetric subset $S \subseteq A$ that doesn't contain the identity element of A . The following two related problems (for $A = \mathbb{R}^n$ or $A = \mathbb{Z}^n$, and various choices of S and Ω) have been the subject of many works:

- 1) (*Finding configurations*) Suppose $\Omega \subseteq A$ has positive upper density, that is, Ω contains a definite proportion of an increasing sequence of \mathbf{d} -balls. When does the equation $x - y = s$ have a solution with $x, y \in \Omega$ and $s \in S$?
- 2) (*Efficient coloring*) Let $\text{Cay}(A, S)$ denote the Cayley graph of A with respect to S . When is the chromatic (or Borel chromatic) number of $\text{Cay}(A, S)$ finite? When yes, establish non-trivial lower and upper bounds.

Let us review some of the existing literature on these themes. Furstenberg, Katznelson and Weiss [17] used tools from ergodic theory to prove that if the upper density of $\Omega \subseteq \mathbb{R}^2$ is strictly positive then for all sufficiently large real numbers ℓ one can find points $x, y \in \Omega$ with euclidean distance ℓ from

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each other i.e., $x - y \in \ell \mathbf{S}^1$ for any large enough ℓ . This subject, known as *Geometric Ramsey theory*, has been the source of many investigations; we refer the reader to Ziegler [32] for a development of this theory. Falconer and Marstrand [14] obtained another proof of Furstenberg, Katznelson and Weiss' theorem by means of geometric measure theory. Subsequently, Bourgain [8] applied ideas from harmonic analysis to generalize the theorem to \mathbb{R}^m . A probabilistic proof is also given recently by Quas [22]. Moreover, a quantitative version of these results has been obtained by Bukh [9] and Oliveira and Vallentin [11]. The euclidean norm is obtained by the unit ball and one might wonder what would happen if the unit ball is replaced with a convex body \mathcal{K} . Notice that \mathcal{K} produces a new norm defined by

$$\|x\|_{\mathcal{K}} := \sup\{t \geq 0 : tx \in \mathcal{K}\}.$$

This problem has been considered by Kolountzakis [19] who proved a result analogues to the theorem of Furstenberg, Katznelson, Weiss, Falconer, Marstrand and Bourgain when \mathcal{K} is *essentially* not a polytope. We refer the reader to the original paper for more details.

Similar problems, involving polynomials, have been studied for $A = \mathbb{Z}$. A theorem of Furstenberg and Sárközy states that if $p(x) \in \mathbb{Z}[x]$ and $p(0) = 0$, then in any set of positive upper density one can find distinct elements x, y in the given set such that the equation

$$x - y = p(n),$$

has a solution for some $n \in \mathbb{Z}$. Bergelson and Leibman [4] (extending on the ideas of Furstenberg) established a far-reaching qualitative generalization of Furstenberg-Sárközy's theorem, often referred to as the Polynomial Szemerédi Theorem. This theorem has been further extended to more general polynomials in [5]. Subsequently Lyall and Magyar [21] obtained a quantitative version of Furstenberg-Sárközy's theorem. More precisely, for $A \subseteq [1, N] \cap \mathbb{Z}$ and a family of linearly independent polynomials $p_1(x), \dots, p_m(x)$ in $\mathbb{Z}[x]$ with $p_i(0) = 0$, Lyall and Magyar proved that if

$$(1) \quad \{p_1(d), \dots, p_m(d)\} \not\subseteq A - A,$$

for any $0 \neq d \in \mathbb{Z}$, then

$$(2) \quad \frac{|A|}{N} \leq C \left(\frac{(\log \log N)^2}{\log N} \right)^{1/m(k-1)}, \quad k := \max_{1 \leq i \leq m} \deg p_i,$$

for some absolute constant $C = C(p_1, \dots, p_m)$. This result is a direct consequence of higher dimensional analogue of Furstenberg-Sárközy's theorem [21, Theorem 3]. For a qualitative version of this result, obtained by ergodic theoretical method, see [6, Corollary 4.2.1]

Let us now turn to the second theme. The most famous question of this type asks about the chromatic number of the unit distance graph, the Cayley graph of \mathbb{R}^m with respect to the unit sphere. In dimension two, this is known as the Hadwiger-Nelson problem and has a long history. It has been known for a long time that

$$4 \leq \chi(\text{Cay}(\mathbb{R}^2, \mathbf{S}^1)) \leq 7.$$

The upper bound is proved via a hexagonal partitioning of the plane and the upper bound can be established by considering a specific 7-vertex subgraph known as Moser spindle. Answering affirmatively a question of Erdős, Frankl and Wilson [16] proved that for $m \geq 3$,

$$c_1^m \leq \chi(\text{Cay}(\mathbb{R}^m, \mathbf{S}^{m-1})).$$

for $c_1 = 1.207 + o(1)$. This was subsequently improved to $c_1 = 1.239 + o(1)$ by Raigorodskii [23]. Larman and Rogers [20] proved the upper bound $\chi(\text{Cay}(\mathbb{R}^m, \mathbf{S}^{m-1})) \leq c_2^m$, for $c_2 = 3 + o(1)$. For more regarding this problem, we refer the reader to [27, 29].

Let A and S be as above, and $\mathcal{G} = \text{Cay}(A, S)$ denote the Cayley graph of A with respect to S . The **Borel chromatic number** of \mathcal{G} , denoted by $\chi_{\text{Bor}}(\mathcal{G})$, is the least cardinal c such that the set A can be partitioned into c Borel subsets none of which contains two adjacent vertices of \mathcal{G} . One of the advantages of working with the Borel chromatic number is that it allows for analytic tools to be deployed. The study of the measurable chromatic number was initiated by Falconer [13] who proved that

$$\chi_{\text{Bor}}(\text{Cay}(\mathbb{R}^m, \mathbf{S}^{m-1})) \geq m + 3.$$

Falconer's method is based on geometric measure theory. Recently, by bringing new ideas from Fourier analysis and linear programming method, Bachoc, Nebe, Oliveira and Vallentin [2], among

other things, extended significantly the lower bound of $\chi_{\text{Bor}}(\text{Cay}(\mathbb{R}^m, \mathbf{S}^{m-1}))$ when $10 \leq m \leq 24$. These bounds have been further extended by Oliveira and Vallentin [11] to the range $3 \leq m \leq 24$.

Another variation of the unit distance graph is the following graph considered by Székely [29]. Let \mathcal{H} be a set of positive real numbers. Székely introduced the graph $\mathcal{G}_{\mathcal{H}}$ with the vertex set \mathbb{R}^m in which two points are joined if their euclidean distance is in \mathcal{H} and conjectured that if \mathcal{H} is not bounded from above then the Borel chromatic number is infinite. Note that the theorem of Furstenberg, Katznelson and Weiss [17] mentioned above answers this question positively.

One of the technique for lower-bounding the chromatic number of finite graphs involves spectral analysis of the graph. The classical result in this direction is due to Hoffman who gave a lower bound for the chromatic number of a finite graph \mathcal{G} in terms of the smallest and largest eigenvalues of the adjacency matrix of \mathcal{G} . Steinhardt [28], by providing a version of Hoffman's spectral bound for infinite topological graphs, was able to give a spectral proof of infiniteness of the Borel chromatic number of *odd distance graph* which is defined by $\text{Cay}(\mathbb{R}^2, \cup_{k \geq 0} (2k+1)\mathbf{S}^1)$. We remark that the odd distance graph, is a special case of Székely's construction when \mathcal{H} is the set of all positive odd integers. Steinhardt's method has been generalized systematically by Bachoc, DeCorte, Oliveira and Vallentin [1]. Notice that for any topological graph \mathcal{G} the obvious bound $\chi(\mathcal{G}) \leq \chi_{\text{Bor}}(\mathcal{G})$ holds. To indicate the power of analytic tools, let us mention that it is an open problem whether the ordinary chromatic number of the odd distance graph is infinite.

All the graphs described so far are Cayley graphs with respect to a family of spheres of various radii. Hence these graphs are defined by algebraic relations. In our previous work [3], by applying the spectral method developed by Bachoc, DeCorte, Oliveira and Vallentin [1], we showed that the Borel chromatic number of the $\text{Cay}(\mathbb{K}^2, \Sigma) = \aleph_0$, where \mathbb{K} is a local field of characteristic zero and $\Sigma = \{(x, y) \in \mathbb{K}^2 : xy = 1\}$. As an application we proved that if $\text{SL}_2(\mathbb{K})$, is partitioned into finitely many Borel sets then in at least one of these sets one can find two matrices A, B such that $\det(A + B) = 0$. Motivated by this, it is natural to ask for a classification of real and p -adic algebraic sets in \mathbb{R}^m or \mathbb{Q}_p^m in which the Borel chromatic number of their associated Cayley graphs is infinite.

Question 1. Let \mathbb{K} be either \mathbb{R} or \mathbb{Q}_p . Classify those Zariski closed sets $V \subseteq \mathbb{A}_{\mathbb{K}}^m$ such that

$$(3) \quad \chi_{\text{Bor}}(\text{Cay}(\mathbb{K}^m, \pm V)),$$

is infinite.

Our interest in this paper is to investigate Question 1. We will develop techniques for establishing “almost non-negativity” bounds for certain oscillatory integrals with polynomial phase and subsequently apply these bounds to combinatorial problems of the types indicated above. As a result, pertaining to *finding configuration* and *effective coloring*, we will obtain non-trivial upper bounds for the independence density of certain Cayley graphs and lower bounds for their Borel chromatic number. Interestingly enough, these results are not generally true even for real analytics curves.

2. MAIN RESULTS

Notation. Through this paper we let \mathbb{K} denote \mathbb{R} or the field of p -adic numbers \mathbb{Q}_p . The vector spaces \mathbb{R}^m and \mathbb{Q}_p^m are respectively equipped with the euclidean and the supremum norm, both of which will be denoted by $\|\cdot\|$. The ball of radius r centered at zero with respect to the metric induced by these norms is denoted by B_r . As a locally compact topological group \mathbb{K}^m has a Haar measure. The measure of a Borel set $A \subseteq \mathbb{K}^m$ will be denoted by $|A|$. When $\mathbb{K} = \mathbb{Q}_p$, the measure is normalized so that \mathbb{Z}_p^m , the ring of p -adic integers, has measure one. For a set $I \subseteq \mathbb{K}^m$, the difference set $I - I$ consists of all differences $i_1 - i_2$ with $i_1, i_2 \in I$.

Before stating the results, let us review some terminology. Let $\mathcal{S} \subseteq \mathbb{K}^m$ be an arbitrary Borel set. The Cayley graph of \mathbb{K}^m with respect to the symmetrized set $\pm\mathcal{S}$, denoted by

$$\mathcal{G} := \text{Cay}(\mathbb{K}^m, \pm\mathcal{S}),$$

is an infinite graph with vertex set \mathbb{K}^m and the edge set $\{\{x, x \pm s\} : x \in \mathbb{K}^m, s \in \mathcal{S}\}$. Note that the vertex set of \mathcal{G} is also furnished with a Borel structure induced by the topology of \mathbb{K} .

An **independent set** of \mathcal{G} is a *Borel subset* of \mathbb{K}^m which does not contain a pair of adjacent vertices. The upper density of any Borel set $I \subseteq \mathbb{K}^m$ is defined by

$$(4) \quad \bar{d}(I) := \limsup_{r \rightarrow \infty} \frac{|I \cap B_r|}{|B_r|}.$$

The **independence ratio** of \mathcal{G} is defined by

$$(5) \quad \bar{\alpha}(\mathcal{G}) := \sup \{ \bar{d}(I) : I \text{ is an independent set of } \mathcal{G} \}.$$

Theorem 2.1. *Let $f_1, \dots, f_m \in \mathbb{K}[x]$ be $m \geq 2$ polynomials such that $1, f_1, \dots, f_m$ are \mathbb{K} -linearly independent. Then there exists constants $C > 0$ and $a_0 > 0$ depending only on f_1, \dots, f_m , such that for any $a > a_0$ and any $T > a$ we have*

$$(6) \quad \bar{\alpha}(\text{Cay}(\mathbb{K}^m, \mathcal{S}_T)) \leq \frac{C}{T - a},$$

where

$$\mathcal{S}_T = \mathcal{S}_{a,T,f_1,\dots,f_m} := \{ \pm(f_1(s), \dots, f_m(s)) \in \mathbb{K}^m : s \in \mathbb{K}, \quad \mathbf{e}^a \leq \|s\| \leq \mathbf{e}^T \};$$

here, $\mathbf{e} = e$, the Euler number, when $\mathbb{K} = \mathbb{R}$ and $\mathbf{e} = p$ when $\mathbb{K} = \mathbb{Q}_p$.

We always assume T and a are integers when $\mathbb{K} = \mathbb{Q}_p$.

Remark 2.2. By a theorem of Steinhaus, the difference set of a set of positive measure contains an open neighborhood of zero. From this it follows immediately that if zero is in the topological closure of \mathcal{S}_T then $\bar{\alpha}(\text{Cay}(\mathbb{K}^m, \pm\mathcal{S}_T)) = 0$. Hence in Theorem 2.1 we must pick a_0 large enough so that $0 \notin \overline{\mathcal{S}_T}$. In this case, one can show that the independence ratio is positive, as will be explained below.

Remark 2.3. Let $f_1(0) = \dots = f_m(0) = 0$, and assume that $(f_1(t), \dots, f_m(t))$ is non-zero for $t \neq 0$. Denote $\mathcal{S}'_T = \{ \pm(f_1(s), \dots, f_m(s)) : e^{-T} \leq s \leq 1 \}$. A careful analysis of the proof of Theorem 2.1 for $\mathbb{K} = \mathbb{R}$, shows that $\bar{\alpha}(\text{Cay}(\mathbb{K}^m, \mathcal{S}'_T)) \ll 1/T$ as $T \rightarrow \infty$. This can be viewed as a quantitative version of the phenomenon observed in Remark 2.2, accounting for the fact that the distance between the set \mathcal{S}'_T and the origin is converging to zero. Moreover, when $f_1(0) = \dots = f_m(0) = 0$, the linear independence condition in Theorem 2.1 can be dropped.

We remark that, when the Borel chromatic of \mathcal{G} is finite, we have the following inequality

$$(7) \quad \chi_{\text{Bor}}(\mathcal{G}) \bar{\alpha}(\mathcal{G}) \geq 1.$$

Fixing a large $a > 0$ and letting $T \rightarrow \infty$, one can see that \mathcal{S}_T is covered by a box of volume $O(\mathbf{e}^{NT})$ for some N . From here one can easily see that $\chi_{\text{Bor}}(\text{Cay}(\mathbb{K}^m, \mathcal{S}_T))$ is bounded from above by $C\mathbf{e}^{NT}$ for some $C, N > 0$ (independent of T). Thus by (7) we obtain $C\mathbf{e}^{-NT} \leq \bar{\alpha}(\text{Cay}(\mathbb{K}^m, \mathcal{S}_T))$. By combining (7) and Theorem 2.1, we obtain the following result.

Corollary 2.4. *With the same notations as Theorem 2.1, we have:*

$$(8) \quad T - a \ll \chi_{\text{Bor}}(\text{Cay}(\mathbb{K}^m, \mathcal{S}_T)) \ll \mathbf{e}^{NT},$$

for some integer N .

Let us describe the key ideas of the proof of Theorem 2.1. First, we use a spectral upper bound for the independence ratio of Cayley graph $\text{Cay}(\mathbb{K}^m, \mathcal{S})$ established in [1] which is analogous to the well-known Hoffman bound for finite graphs. The crucial step in the proof will consist of establishing a **uniform** lower bound for the family of oscillatory real and p -adic integrals that arises as the Fourier transform of a carefully chosen probability measures on \mathcal{S}_T . The core of the proof is to use van der Corput's lemma to obtain a lower bound for these oscillatory integrals which is **independent** of the frequency vector. It turns out that the frequency vectors that lie between two specific affine hyperplanes have to be handled differently from the rest. Eventually, we will find a suitable partition of the domain of integration into a uniformly bounded number of intervals (sphere in the p -adic case), which enables us to apply van der Corput's lemma and obtain a uniform lower bound for the Fourier transform.

Theorem 2.1 can be reformulated in the following form, resembling Furstenberg-Sárközy's theorem.

Corollary 2.5 (High dimensional polynomial configurations). *Let f_1, \dots, f_m, a_0, C and \mathbf{e} be as in Theorem 2.1. For any $T > a > a_0$ and any Borel set $I \subseteq \mathbb{K}^m$ with*

$$\bar{d}(I) > \frac{C}{T - a},$$

there exists $s \in \mathbb{K}$ with $\mathbf{e}^a \leq \|s\| \leq \mathbf{e}^T$ and distinct $\mathbf{x}_1, \mathbf{x}_2 \in I$ such that

$$\mathbf{x}_1 - \mathbf{x}_2 = (f_1(s), \dots, f_m(s)).$$

Remark 2.6. It is worth mentioning that even when f_1, \dots, f_m have integer coefficients, the above theorem is not a consequence of Furstenberg and Sárközy theorem. For instance take $I = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \in [0, 1/4] + 9\mathbb{N}\}$, which has positive density in \mathbb{R}^2 . Let $f_1(x) = x$ and $f_2(x) = x^2 + 1$. Observe that $I \cap \mathbb{Z}^2$ has positive upper density in \mathbb{Z}^2 since $(3\mathbb{N})^2 \subseteq I \cap \mathbb{Z}^2$. Due to arithmetic obstructions, there are no $\mathbf{x}_1, \mathbf{x}_2 \in I \cap \mathbb{Z}^2$ and $d \in \mathbb{Z}$ such that $\mathbf{x}_1 - \mathbf{x}_2 = (f_1(d), f_2(d))$.

Corollary 2.7 (Polynomial configurations). *Let f_1, \dots, f_m, a_0, C and \mathbf{e} be as in Theorem 2.1. For any $T > a > a_0$ and any Borel set $I \subseteq \mathbb{K}$ with*

$$\bar{d}(I) > \left(\frac{C}{T - a} \right)^{1/m},$$

there exists $s \in \mathbb{K}$ with $\mathbf{e}^a \leq \|s\| \leq \mathbf{e}^T$ such that

$$\{f_1(s), \dots, f_m(s)\} \subseteq I - I.$$

Proof. Let I be as above. Then $\bar{d}(I^m) > C/(T - a)$ and so by applying Corollary 2.5 we obtain the result. \square

We now discuss the Borel chromatic number of Cayley graphs assigned to algebraic varieties.

Remark 2.8. By Steinhaus theorem, mentioned above, if zero is in the topological closure of $\mathcal{S} \subseteq \mathbb{K}^m$ then $\chi_{\text{Bor}}(\text{Cay}(\mathbb{K}^m, \pm\mathcal{S})) > \aleph_0$. Conversely, if $0 \notin \bar{\mathcal{S}}$, then the Borel chromatic number of $\text{Cay}(\mathbb{K}^m, \pm\mathcal{S})$ is at most \aleph_0 . Moreover, one can show that $\chi_{\text{Bor}}(\text{Cay}(\mathbb{K}^m, \pm\mathcal{S}))$ is finite when \mathcal{S} is compact and $0 \notin \bar{\mathcal{S}}$.

From Corollary 2.4 and Remark 2.8 we deduce the following qualitative result.

Corollary 2.9 (Infinite Borel chromatic number). *Let f_1, \dots, f_m be as in Theorem 2.1. Then for any large enough $a > 0$ we have*

$$(9) \quad \chi_{\text{Bor}}(\text{Cay}(\mathbb{K}^m, \mathcal{S})) = \aleph_0,$$

where

$$\mathcal{S} = \mathcal{S}_a = \{\pm(f_1(s), \dots, f_m(s)) \in \mathbb{K}^m : s \in \mathbb{K}, \quad a \leq \|s\|\}.$$

The assumption that $1, f_1, \dots, f_m$ are linearly independent is quite natural in this context. In fact if $1, f_1, \dots, f_m$ are linearly dependent, then \mathcal{S} will be contained in an affine hyperplane in \mathbb{K}^m and one can easily see that the Borel chromatic number of the Cayley graph associated to a hyperplane which does not pass through the origin is finite.

The next theorem gives a partial answer to Question 1. Indeed, it shows that the Borel chromatic number of the Cayley graph assigned to real or p -adic varieties that are parameterized by polynomials, is infinite. Notice that by Remark 2.8, we need to remove a neighbourhood of the origin from the algebraic set V in order to have a non-trivial answer for Question 1.

Theorem 2.10 (Multivariate polynomial configurations). *Let $F_1, \dots, F_m \in \mathbb{K}[x_1, \dots, x_d]$ be $m \geq 2$ polynomials such that $1, F_1, \dots, F_m$ are linearly independent. Then for any $\delta > 0$ we have*

$$(10) \quad \chi_{\text{Bor}}(\text{Cay}(\mathbb{K}^m, \mathcal{S}_\delta)) = \aleph_0,$$

where

$$\mathcal{S}_\delta = \{\pm(F_1(\mathbf{s}), \dots, F_m(\mathbf{s})) \in \mathbb{K}^m : \mathbf{s} \in \mathbb{K}^d\} \setminus B_\delta.$$

Proof. Suppose that the maximum degree of F_1, \dots, F_m is $\ell - 1$. Set $n_i = \ell^{i-1}$ for all $1 \leq i \leq d$ and substitute x_i with t^{n_i} . Thus the monomial $x_1^{\alpha_1} \dots x_d^{\alpha_d}$ will be substituted by $t^{h(\alpha_1, \dots, \alpha_d)}$, where $h(\alpha_1, \dots, \alpha_d) = \sum_{i=1}^d \alpha_i \ell^{i-1}$. Notice that h is injective on the cube defined by $0 \leq \alpha_i \leq \ell - 1$. This

clearly follows from the fact that $\alpha_1, \dots, \alpha_d$ are digits of $h(\alpha_1, \dots, \alpha_d)$ when expressed in base ℓ , and hence are uniquely determined by $h(\alpha_1, \dots, \alpha_d)$. Thus we obtain $m \geq 2$ polynomials

$$f_i(t) := F_i(t^{n_1}, \dots, t^{n_d}),$$

such that $1, f_1, \dots, f_m$ are \mathbb{K} -linearly independent. Then by applying Corollary 2.9 we obtain the result. \square

Recall that the clique number of a graph \mathcal{G} , denoted by $\omega(\mathcal{G})$, is the largest n for which \mathcal{G} has a subgraph isomorphic to the complete graph K_n . It is clear that $\omega(\mathcal{G}) \leq \chi_{\text{Bor}}(\mathcal{G})$. Corollary 2.9 would be trivial if one could exhibit arbitrarily large cliques in \mathcal{G} . However, we will show that there is an algebraic obstruction to this, suggesting that the infinitude of the Borel chromatic number cannot be proven by such local arguments. Obviously, it is enough to prove the nonexistence of large cliques when \mathbb{K} is replaced by an algebraic closure.

Theorem 2.11. *Let K be an algebraic closed field of characteristic 0, and let $V \subseteq K^m$ be an irreducible variety with $\dim V = 1$ that is not an affine line. Then*

$$(11) \quad \omega(\text{Cay}(K^m, \pm V)) < \infty.$$

Another point that has to be addressed is the extent to which the polynomial nature of the map is important for the above results to hold. We will show:

Theorem 2.12. *There exists a real analytic curve $f : (a, b) \rightarrow \mathbb{R}^2$ such that the image of f does not lie between any two parallel lines and the Borel chromatic number of the Cayley graph of \mathbb{R}^2 with respect to the image of f is finite.*

As indicated above, proving statements analogous to Theorem 2.10 for the ordinary chromatic number seems to be quite difficult. This is essentially due to the fact that proving such an statement is equivalent to finding (or proving the existence of) *finite* subgraphs of arbitrarily large chromatic number inside the Cayley graphs of \mathbb{K}^n . In the last section of this paper, we will prove a general theorem that highlights this point by showing the question of determining the chromatic number over \mathbb{C} is indeed algebraic in nature. This theorem (and the techniques used in its proof) are independent of the rest of the paper, and is solely included for the sake of comparison.

Let V be an algebraic set defined over \mathbb{Q} given by polynomials $F_i \in \mathbb{Q}[x_1, \dots, x_m]$ for $1 \leq i \leq n$. Without loss of generality, we can assume that $F_i \in \mathbb{Z}[x_1, \dots, x_m]$. If R is any ring of characteristic zero, we can define

$$V(R) = \{(x_1, \dots, x_m) \in R^m : F_i(x_1, \dots, x_m) = 0, \quad 1 \leq i \leq n\}.$$

For p prime, we will denote by \mathbb{Z}_p the ring of p -adic integers.

Theorem 2.13. *Let V be an irreducible variety defined over \mathbb{Q} . Then*

$$(12) \quad \chi(\text{Cay}(\mathbb{C}^m, \pm V(\mathbb{C}))) = \sup_p \chi(\text{Cay}(\mathbb{Z}_p^m, \pm V(\mathbb{Z}_p))),$$

if one of the two sides is finite. The sup is taken over all primes p .

The proof of Theorem 2.13 relies on a theorem of de Bruijn and Erdős which relates the chromatic number of a graph to its finite subgraphs and an embedding theorem of Cassels.

3. PRELIMINARIES

We begin by recalling some basic facts on Borel chromatic number. Let us first introduce some notation. Throughout the paper we let $\mathcal{S} \subset \mathbb{K}^m$ denote a symmetric set, i.e., $-\mathcal{S} = \mathcal{S}$, and $0 \notin \mathcal{S}$. We define the Cayley graph $\mathcal{G} := \text{Cay}(\mathbb{K}^m, \mathcal{S})$ to be the graph with the vertex set \mathbb{K}^m , in which two vertices $x, y \in \mathbb{K}^m$ are adjacent if and only if $x - y \in \mathcal{S}$. Recall that $I \subseteq \mathbb{K}^m$ is called an independent set of \mathcal{G} if $(I - I) \cap \mathcal{S} = \emptyset$.

Lemma 3.1. *With the above notations, $\chi_{\text{Bor}}(\mathcal{G}) \leq \aleph_0$ if and only if $0 \notin \bar{\mathcal{S}}$.*

Proof. First assume that $0 \notin \bar{\mathcal{S}}$. Choose $\delta > 0$ such that $B_{2\delta} \cap \mathcal{S} = \emptyset$. This, in particular, implies that B_δ is an independent set for \mathcal{G} . Note that since \mathbb{Q}^m is dense in \mathbb{K}^m , we have $\mathbb{K}^m = \cup_{v \in \mathbb{Q}^m} (v + B_\delta)$. This countable cover can then be easily transformed further into a **disjoint** countable cover of independent Borel (in fact, locally closed) sets, yielding the desirable coloring.

Now assume $\chi_{\text{Bor}}(\mathcal{G}) \leq \aleph_0$. From countable additivity, it follows that there exists a color class I with positive Lebesgue measure. Using a theorem of Steinhaus (whose proof works word-for-word in the p -adic case), there exists $\delta_1 > 0$ such that $B_{\delta_1} \subseteq (I - I)$. Since I is independent, it follows that $\mathcal{S} \cap B_{\delta_1} = \emptyset$, and so $0 \notin \bar{\mathcal{S}}$. \square

If, furthermore, \mathcal{S} is a bounded Borel set, then the Borel chromatic number of \mathcal{G} is finite.

Lemma 3.2. *Let \mathcal{S} be a bounded Borel set such that $0 \notin \bar{\mathcal{S}}$. Then $\chi_{\text{Bor}}(\mathcal{G})$ is finite.*

Proof. See [3, Lemma 3.3]. \square

Since invertible linear maps preserve the additive structure, the following lemma is immediate.

Lemma 3.3. *Let $B \in \text{GL}_m(\mathbb{K})$ be an invertible matrix. Then for any symmetric set $\mathcal{S} \subseteq \mathbb{K}^m$ we have*

$$\chi_{\text{Bor}}(\text{Cay}(\mathbb{K}^m, \mathcal{S})) = \chi_{\text{Bor}}(\text{Cay}(\mathbb{K}^m, B(\mathcal{S}))).$$

Moreover, the same is true for the ordinary chromatic number.

4. A SPECTRAL BOUND FOR INDEPENDENCE RATIO

Let $\mathcal{G} = (V, E)$ be a finite regular graph with n vertices and the adjacency matrix A . Let

$$\lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_{n-1},$$

denote the spectrum of A and assume that $I \subseteq V$ is an independent set of \mathcal{G} . The celebrated Hoffman bound states that

$$(13) \quad \frac{|I|}{n} \leq -\frac{\lambda_{n-1}}{\lambda_0 - \lambda_{n-1}} = -\frac{\lambda_{\min}}{\lambda_{\max} - \lambda_{\min}}.$$

This yields the following upper bound for the chromatic number:

$$(14) \quad \chi(\mathcal{G}) \geq -\frac{\lambda_{\max} - \lambda_{\min}}{\lambda_{\min}}.$$

It is a reinterpretation of this inequality that is the key to the generalization we will need later. Denote by $L^2(V)$ the Hilbert space of all complex-valued functions on V , equipped with the inner product $\langle f, g \rangle := \sum_{v \in V} f(v) \overline{g(v)}$. The adjacency operator $A : L^2(V) \rightarrow L^2(V)$, defined by $Af(v) = \sum_{vw \in E} f(w)$, is easily seen to be self-adjoint. Further, one can see that its *numerical range* defined by

$$W(A) := \{\langle Af, f \rangle : \|f\|_2 = 1\},$$

is equal to $[\lambda_{n-1}, \lambda_0]$. It is a routine verification that a subset $I \subseteq V$ is independent in the graph \mathcal{G} if and only if for all $f \in L^2(V)$ supported on I , one has $\langle Af, f \rangle = 0$. This novel interpretation of independent sets is used in [1] to prove an analog of the Hoffman bound for certain Cayley graphs of the Euclidean additive group \mathbb{R}^n . As we will need to work in a slightly more general framework, it will be useful to briefly review some of the key points of [1]; for details, we refer the reader to the original paper [1].

Abstractly, let (V, Σ, μ) be a **probability space**, consisting of a set V , a σ -algebra Σ on V , and a probability measure μ , and consider the Hilbert space $L^2(V)$ of square integrable functions with respect to the inner product:

$$\langle f, g \rangle = \int_V f(x) \overline{g(x)} d\mu(x).$$

For a bounded and self-adjoint operator $A : L^2(V) \rightarrow L^2(V)$, one can show that the *numerical range* of A defined by

$$W(A) = \{\langle Af, f \rangle : \|f\|_2 = 1\},$$

is an interval in \mathbb{R} . We denote the endpoints of $W(A)$ by

$$m(A) := \inf\{\langle Af, f \rangle : \|f\|_2 = 1\}, \quad M(A) := \sup\{\langle Af, f \rangle : \|f\|_2 = 1\}.$$

Definition 4.1. Let $A : L^2(V) \rightarrow L^2(V)$ be a bounded, self-adjoint operator. A measurable set $I \subseteq V$ is called an **independent set for A** if $\langle Af, f \rangle = 0$ for each $f \in L^2(V)$ which vanishes almost everywhere outside of I . Moreover the chromatic number of A , denoted by $\chi(A)$, equals the least number k such that one can partition V into k independent sets for A .

The **independence ratio** of A is defined by

$$(15) \quad \bar{\alpha}(A) := \sup\{\mu(I) : I \text{ is an independent set of } A\}.$$

The following theorem—which can be obtained by a clever modification of the proof of Hoffmann’s bound presented by Bollobás [7, Chapter VIII.2]—recovers (13) when A is the adjacency matrix of a finite regular graph.

Theorem 4.2. Let (V, Σ, μ) be a probability space and let $A : L^2(V) \rightarrow L^2(V)$ be a nonzero, bounded, self-adjoint operator. Fix a real number R and set $\varepsilon = \|A\mathbf{1} - R\mathbf{1}\|_2$, where $\mathbf{1} = \mathbf{1}_V$ is the characteristic function of V . Suppose there exists a set $I \subseteq V$ with $\mu(I) > 0$ which is independent for A . Then, if $R - m(A) - \varepsilon > 0$, we have

$$(16) \quad \bar{\alpha}(A) \leq \frac{-m(A) + 2\varepsilon}{R - m(A) - \varepsilon}.$$

Proof. See [1, Theorem 2.2]. □

Moreover, Bachoc, DeCortede, Oliveira and Vallentin proved [1, Theorem 2.3] the following theorem which is an analogue of (14)

Theorem 4.3. Let $A : L^2(V) \rightarrow L^2(V)$ be a nonzero, bounded and self-adjoint operator. Moreover assume that $\chi(A) < \infty$. Then

$$(17) \quad \chi(A) \geq 1 - \frac{M(A)}{m(A)}.$$

Remark 4.4. Similar to the Hoffman bound, when $A \neq 0$ and $\chi(A) < \infty$ the proof of the above theorem shows that $m(A) < 0$ and $M(A) > 0$.

We now apply these theorems to the Borel chromatic number and independence ratio of Cayley graphs with the vertex set \mathbb{K}^m . All that follows is parallel to [1], where the case $\mathbb{K} = \mathbb{R}$ is dealt with; in fact the arguments in [1] can be easily seen to work also in the p -adic case. For the convenience of the reader we will briefly sketch the key points.

Let us denote by dx a Haar measure on \mathbb{K} . When $\mathbb{K} = \mathbb{Q}_p$, we assume that it is normalized such that $\int_{\mathbb{Z}_p} dx = 1$. We will also write dx for the product (Haar) measure $dx_1 \cdots dx_m$ on \mathbb{K}^m . For a Borel set $E \subseteq \mathbb{K}^m$ we use $|E| = \int_E dx$ to denote the measure of E . As before, $\mathcal{S} \subseteq \mathbb{K}^m$ is a symmetric (i.e., $-\mathcal{S} = \mathcal{S}$), Borel measurable set (with respect to the σ -algebra generated by the open sets), which does not contain the origin in its closure.

Throughout this section, we let μ be a **probability Borel measure** on \mathbb{K}^m supported in \mathcal{S} . Therefore μ is a Radon measure [15, Theorem 7.8]. We will also assume that μ is symmetric, i.e., $\mu(-\mathcal{A}) = \mu(\mathcal{A})$ holds for all Borel measurable sets \mathcal{A} . Define the bounded, self-adjoint operator A_μ by

$$A_\mu : L^2(\mathbb{K}^m) \rightarrow L^2(\mathbb{K}^m), \quad f \mapsto f * \mu,$$

where

$$(18) \quad f * \mu(x) = \int_{\mathbb{K}^m} f(x - y) d\mu(y).$$

From now on, assume that \mathcal{S} is also bounded. By Lemma 3.2, the Borel chromatic number of $\text{Cay}(\mathbb{K}^m, \mathcal{S})$ is finite. Since $\text{Supp}(\mu) \subseteq \mathcal{S}$, any independent set I for $\text{Cay}(\mathbb{K}^m, \mathcal{S})$ is also an independent set for A_μ . Therefore by Theorem 4.3 we have

$$(19) \quad 1 - \frac{M(A_\mu)}{m(A_\mu)} \leq \chi(A_\mu) \leq \chi_{\text{Bor}}(\text{Cay}(\mathbb{K}^m, \mathcal{S})).$$

The numerical range of A_μ can now be determined using Fourier analysis. Below, we will review a number of basic properties of the Fourier transform over \mathbb{R} and \mathbb{Q}_p . For details we refer the reader to [25, 30]. For $\mathbb{K} = \mathbb{R}$, we will use the character

$$\psi : \mathbb{R} \rightarrow \mathbb{C}^*, \quad \psi(x) = \exp(2\pi i x).$$

When $\mathbb{K} = \mathbb{Q}_p$, for $x \in \mathbb{Q}_p$, we write n_x for the smallest non-negative integer such that $p^{n_x}x \in \mathbb{Z}_p$. Let $r_x \in \mathbb{Z}$ be such that $r_x \equiv p^{n_x}x \pmod{p^{n_x}}$. It is well-known that the following map (called the Tate character)

$$(20) \quad \psi : \mathbb{Q}_p \rightarrow \mathbb{C}^*, \quad \psi(x) = \exp\left(\frac{2\pi i r_x}{p^{n_x}}\right),$$

is a non-trivial character of $(\mathbb{Q}_p, +)$ with the kernel \mathbb{Z}_p .

The Fourier transforms of $f \in L^1(\mathbb{K}^m)$ and μ are respectively defined by the integrals

$$\widehat{f}(u) = \int_{\mathbb{K}^m} \overline{\psi}(x \cdot u) f(x) dx, \quad \widehat{\mu}(u) = \int_{\mathbb{K}^m} \overline{\psi}(x \cdot u) d\mu(x).$$

Here $x \cdot u$ is the standard bilinear form on \mathbb{K}^m , and $\overline{\psi}$ is the complex conjugate of ψ . We remark that $\widehat{\mu}$ is a real-valued function since μ is symmetric. By Plancherel's theorem the Fourier transform extends to an isometry on $L^2(\mathbb{K}^m)$ and thus for any $f \in L^2(\mathbb{K}^m)$ we have

$$(21) \quad \langle A_\mu f, f \rangle = \langle \widehat{A_\mu f}, \widehat{f} \rangle = \langle \widehat{f * \mu}, \widehat{f} \rangle = \langle \widehat{\mu} \widehat{f}, \widehat{f} \rangle.$$

Lemma 4.5. *Let μ be a symmetric probability Borel measure on \mathbb{K}^m with its support contained in \mathcal{S} . Then the numerical range of A_μ is given by*

$$m(A_\mu) = \inf_{u \in \mathbb{K}^m} \widehat{\mu}(u), \quad 1 = M(A_\mu) = \sup_{u \in \mathbb{K}^m} \widehat{\mu}(u).$$

Proof. From (21) combined with the fact that the Fourier transform on $L^2(\mathbb{K}^m)$ is an isometry, we deduce that the numerical range of the operator A_μ is the same as the numerical range of the multiplication operator $g \mapsto \widehat{\mu}g$. Since μ is a symmetric probability Borel measure, $\widehat{\mu}$ is a bounded continuous real-valued function. Now let $g \in L^2(\mathbb{K}^m)$ with $\|g\|_2 = 1$. Evidently

$$\langle \widehat{\mu}g, g \rangle = \int_{\mathbb{K}^m} \widehat{\mu}(x) |g(x)|^2 dx \geq \inf_{u \in \mathbb{K}^m} \widehat{\mu}(u),$$

and so $m(A_\mu) \geq \inf_{u \in \mathbb{K}^m} \widehat{\mu}(u)$. Now let $\varepsilon > 0$ and pick $x_0 \in \mathbb{K}^m$ with $\widehat{\mu}(x_0) \leq \inf_{u \in \mathbb{K}^m} \widehat{\mu}(u) + \varepsilon$. Let $B_\delta = B_\delta(x_0)$ be the ball of radius δ centered at x_0 . Since $\widehat{\mu}$ is continuous, we conclude that

$$\lim_{\delta \rightarrow 0} \int_{\mathbb{K}^m} \widehat{\mu}(x) \left| \frac{\mathbf{1}_{B_\delta}(x)}{\sqrt{|B_\delta|}} \right|^2 dx = \lim_{\delta \rightarrow 0} \frac{1}{|B_\delta|} \int_{B_\delta} \widehat{\mu}(x) dx = \widehat{\mu}(x_0),$$

where $\mathbf{1}_{B_\delta}$ is the characteristic function of B_δ . Hence

$$m(A_\mu) \leq \widehat{\mu}(x_0) \leq \inf_{u \in \mathbb{K}^m} \widehat{\mu}(u) + \varepsilon,$$

and therefore we have $m(A_\mu) = \inf_{u \in \mathbb{K}^m} \widehat{\mu}(u)$. Similarly, since μ is a probability measure, we have $M(A_\mu) = \sup_{u \in \mathbb{K}^m} \widehat{\mu}(u) = 1$. \square

We can now summarize these in the following theorem.

Theorem 4.6. *Let \mathbb{K} be either \mathbb{R} or a p -adic field \mathbb{Q}_p and let \mathcal{S} be a bounded symmetric Borel measurable subset of \mathbb{K}^m which does not contain the origin in its closure. Then for any symmetric, probability Borel measure μ on \mathbb{K}^m with its support contained in \mathcal{S} we have*

$$(22) \quad 1 - \frac{1}{\inf_{u \in \mathbb{K}^m} \widehat{\mu}(u)} \leq \chi_{\text{Bor}}(\text{Cay}(\mathbb{K}^m, \mathcal{S})).$$

Let us now consider the independence ratio of $\text{Cay}(\mathbb{K}^m, \mathcal{S})$. For $r > 0$, let $B_r \subseteq \mathbb{K}^m$ be the ball of radius r centered at the origin, and normalize the induced measure on B_r . Define

$$A_\mu^r : L^2(B_r) \rightarrow L^2(B_r),$$

by

$$A_\mu^r(f) := (A_\mu \bar{f})|_{B_r},$$

where \bar{f} is the extension of f to \mathbb{K}^m defined to be zero on $\mathbb{K}^m \setminus B_r$. By abuse of notation, we will continue to write f for \bar{f} . As mentioned before, any independent set I of $\text{Cay}(\mathbb{K}^m, \mathcal{S})$ is also an independent set of A_μ . Hence $I \cap B_r$ is an independent set for the operator A_μ^r . From the definition of the independence ratio of an operator, we obtain the following density bound:

$$\frac{|B_r \cap I|}{|B_r|} \leq \bar{\alpha}(A_\mu^r).$$

This implies that

$$(23) \quad \bar{\alpha}(\text{Cay}(\mathbb{K}^m, \mathcal{S})) \leq \limsup_{r \rightarrow \infty} \bar{\alpha}(A_\mu^r).$$

For a given $r > 0$, define

$$(24) \quad R = R(r) = \langle A_\mu^r \mathbf{1}_{B_r}, \mathbf{1}_{B_r} \rangle_{L^2(B_r)}, \quad \epsilon = \epsilon(r) = \|A_\mu^r \mathbf{1}_{B_r} - R(r) \mathbf{1}_{B_r}\|_{L^2(B_r)},$$

where $\mathbf{1}_{B_r}$ is the characteristic function of B_r .

Lemma 4.7. *With the above notation, we have*

$$(25) \quad \lim_{r \rightarrow \infty} m(A_\mu^r) = m(A), \quad \lim_{r \rightarrow \infty} R(r) = 1, \quad \lim_{r \rightarrow \infty} \epsilon(r) = 0.$$

Proof. Let $f \in L^2(\mathbb{K}^m)$. Define $f^r := f|_{B_r}$. Then

$$\langle A_\mu^r f^r, f^r \rangle_{L^2(B_r)} = \frac{\langle A_\mu f^r, f^r \rangle_{L^2(\mathbb{K}^m)}}{|B_r|} \geq m(A_\mu) \frac{\langle f^r, f^r \rangle_{L^2(\mathbb{K}^m)}}{|B_r|} = m(A_\mu) \langle f^r, f^r \rangle_{L^2(B_r)}.$$

Hence $m(A_\mu^r) \geq m(A_\mu)$. Conversely, as $r \rightarrow \infty$, we have

$$m(A_\mu^r) \leq \frac{\langle A_\mu^r f^r, f^r \rangle_{L^2(B_r)}}{\langle f^r, f^r \rangle_{L^2(B_r)}} = \frac{\langle A_\mu f^r, f^r \rangle_{L^2(\mathbb{K}^m)}}{\langle f^r, f^r \rangle_{L^2(\mathbb{K}^m)}} \rightarrow \frac{\langle A_\mu f, f \rangle_{L^2(\mathbb{K}^m)}}{\langle f, f \rangle_{L^2(\mathbb{K}^m)}},$$

since $f^r \rightarrow f$ in $L^2(\mathbb{K}^m)$. This shows that $\lim_{r \rightarrow \infty} m(A_\mu^r) = m(A_\mu)$.

Recall that μ is a probability measure with $\text{Supp}(\mu) \subseteq \mathcal{S}$. Thus for every $x \in B_r$ we have

$$(26) \quad A_\mu^r \mathbf{1}_{B_r}(x) = \int_{\mathcal{S}} \mathbf{1}_{B_r}(x - y) d\mu(y).$$

\mathcal{S} is bounded, and so we assume that $\mathcal{S} \subseteq B_d$ for some $d > 0$. For $\mathbb{K} = \mathbb{Q}_p$ we get the desired limits since for any $r > d$, by the ultrametric metric inequality, we obtain $A_\mu^r \mathbf{1}_{B_r}(x) = 1$. Therefore assume that $\mathbb{K} = \mathbb{R}$. Let $r > d$ and $x \in B_{r-d}$. Then from (26) we deduce that $A_\mu^r \mathbf{1}_{B_r}(x) = 1$ for all $x \in B_{r-d}$. Hence

$$\lim_{r \rightarrow \infty} R(r) = \lim_{r \rightarrow \infty} \frac{1}{|B_r|} \int_{B_r} A_\mu^r \mathbf{1}_{B_r}(x) dx = 1.$$

Finally

$$\frac{1}{|B_r|} \int_{B_r} |A_\mu^r \mathbf{1}_{B_r}(x) - 1|^2 dx \leq 2 \frac{|B_r \setminus B_{r-d}|}{|B_r|} \rightarrow 0, \quad r \rightarrow \infty.$$

Then by the triangular inequality we deduce that $\lim_{r \rightarrow \infty} \epsilon(r) = 0$. \square

Therefore by applying Theorem 4.2, Lemma 4.5, Lemma 4.7 and (23) we obtain the following theorem:

Theorem 4.8. *Let \mathbb{K} be either \mathbb{R} or a p -adic field \mathbb{Q}_p and let \mathcal{S} be a bounded, symmetric Borel subset of \mathbb{K}^m which does not contain the origin in its closure. Then for any symmetric, probability Borel measure μ on \mathbb{K}^m with its support contained in \mathcal{S} we have*

$$(27) \quad \bar{\alpha}(\text{Cay}(\mathbb{K}^m, \mathcal{S})) \leq -\frac{\inf_{u \in \mathbb{K}^m} \hat{\mu}(u)}{1 - \inf_{u \in \mathbb{K}^m} \hat{\mu}(u)}.$$

5. OSCILLATORY INTEGRALS WITH REAL POLYNOMIAL PHASE

We are now ready to prove Theorem 2.1. Throughout we fix $m \geq 2$ polynomials $f_1(x), \dots, f_m(x)$ with real coefficients such that

$$(28) \quad 1, f_1(x), \dots, f_m(x),$$

are linearly independents over \mathbb{R} . Let $n := \max_i \deg f_i$. From (28) we deduce that $m \leq n$. Let $a_0 = \max\{t \geq 0 : f_1(e^t) = \dots = f_m(e^t) = 0\}$. Fix $a > a_0$, and for $T > a$, consider the following symmetric, bounded closed set

$$\mathcal{S}_T := \mathcal{S}_{a,T,f_1,\dots,f_m} = \{\pm(f_1(s), \dots, f_m(s)) \in \mathbb{R}^m : s \in [e^a, e^T]\}.$$

Since $a > a_0$, we have $0 \notin \overline{\mathcal{S}_T}$. Let μ_T be the Borel measure on \mathbb{R}^m defined for every Borel subset $E \subseteq \mathbb{R}^m$ via

$$\mu_T(E) = \frac{1}{2(T-a)} \int_{e^a}^{e^T} \frac{\mathbf{1}_E(f_1(s), \dots, f_m(s)) + \mathbf{1}_{(-E)}(f_1(s), \dots, f_m(s))}{s} ds.$$

It is easy to verify that μ_T is a probability measure on \mathbb{R}^m and $\text{Supp}(\mu_T) = \mathcal{S}_T$. The Fourier transform of this measure is given by

$$\widehat{\mu}_T(\lambda_1, \dots, \lambda_m) = \frac{1}{(T-a)} \int_{e^a}^{e^T} \frac{\cos(2\pi(\lambda_1 f(s) + \dots + \lambda_m f_m(s)))}{s} ds, \quad (\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m;$$

for our purposes, it would be suitable to apply the logarithmic change of variables $s = e^t$:

$$\widehat{\mu}_T(\lambda_1, \dots, \lambda_m) = \frac{1}{T-a} \int_a^T \cos(2\pi(\lambda_1 f_1(e^t) + \dots + \lambda_m f_m(e^t))) dt.$$

Theorem 5.1. *There exists a constant $C > 0$, depending only on the polynomials f_1, \dots, f_m , such that for all $(\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m$, and any $T > a$ we have*

$$(29) \quad \frac{-C}{T-a} \leq \widehat{\mu}_T(\lambda_1, \dots, \lambda_m).$$

Before we start with the proof of this theorem, we will state and prove two simple combinatorial lemmas. In what follows, two intervals I and J are called disjoint if $I \cap J$ has no interior point.

Lemma 5.2. *Let $A_1, \dots, A_n \subseteq \mathbb{R}$ be such that each A_i is a union of at most n intervals. Then there exist $k \leq 2n^4$ disjoint intervals I_1, \dots, I_k such that*

$$\bigcup_{i=1}^n A_i = \bigcup_{i=1}^k I_i,$$

and for each $1 \leq j \leq k$, there exists $1 \leq i \leq n$ such that $I_j \subseteq A_i$.

Lemma 5.3. *Let $g(t)$ be a polynomial of degree $n \geq 1$ and set $\Phi(t) = g(e^t)$. Then for any $M > 0$, the set $\{t \in \mathbb{R} : |\Phi(t)| \geq M\}$ can be split to a disjoint union of intervals I_1, \dots, I_k such that $k \leq 3n$ and $\Phi'(t)$ is monotone on each I_i .*

Proof. Consider the superlevel set $T_M := \{t \in \mathbb{R} : |\Phi(t)| > M\}$. As T_M is open, it is either \mathbb{R} or a union of open intervals $I_\alpha, 1 \leq \alpha \leq \ell$ with each end-point a solution for $\Phi(t) = \pm M$. Note that since the exponential map is strictly increasing, the equation $\Phi(t) = \pm M$ has at most $2n$ solution, hence, $\ell \leq 2n$. Suppose $[a, b]$ is one of these intervals such that the equation $\Phi''(t) = 0$ has $s \geq 1$ roots $a \leq r_1 < \dots < r_s \leq b$ in $[a, b]$. Upon replacing each such interval $[a, b]$ by the disjoint union $[a, r_1] \cup (r_1, r_2] \cup \dots \cup (r_s, b]$ the number of intervals increases by at most n (a bound for the number of roots of $\Phi''(t) = 0$), while Φ' is monotone over each one of the new intervals. The total number of produced intervals is $n + \ell \leq 3n$. \square

Let $(\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m$ be an arbitrary vector. For the reset of this section we set

$$(30) \quad \Phi(t) := \Phi_{(\lambda_1, \dots, \lambda_m)}(t) = \lambda_1 f_1(e^t) + \dots + \lambda_m f_m(e^t).$$

Let us recall the van der Corput's lemma.

Theorem 5.4 (van der Corput's Lemma). *Suppose that a real-valued function $\psi : (a, b) \rightarrow \mathbb{R}$ satisfies $|\psi^{(k)}(t)| \geq \lambda > 0$ for all $t \in (a, b)$, where $k \geq 1$ is an integer. Then*

$$\left| \int_a^b e^{i\psi(t)} dt \right| \leq \frac{12k}{|\lambda|^{1/k}},$$

provided, in addition when $k = 1$, that $\psi'(t)$ is monotonic on (a, b) .

Proof of Theorem 5.1. Notice that $\widehat{\mu}_T(0, \dots, 0) = 1$, we can assume that $(\lambda_1, \dots, \lambda_m)$ is not-zero. We will prove Theorem 5.1 by considering two cases.

5.1. Low frequency case. Suppose that

$$(31) \quad \left| \sum_{i=1}^m \lambda_i f_i(0) \right| \leq 1/8.$$

Lemma 5.5. *There exist real numbers $\alpha_1, \dots, \alpha_n$, where $n = \max_i \deg f_i$, such that for any vector $(\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m$:*

$$(32) \quad \Phi(t) = \sum_{i=1}^m \lambda_i f_i(0) + \sum_{k=1}^n \alpha_k \Phi^{(k)}(t)$$

Proof. For any $1 \leq i \leq m$, let $f_i(t) = \sum_{j=0}^n a_{ij} t^j$. Using Lagrange interpolation method (or Vandermonde determinant) we can find real numbers $\alpha_1, \dots, \alpha_n$, not all zero, such that for every $1 \leq j \leq n$ the following equality holds:

$$(33) \quad \sum_{k=1}^n \alpha_k j^k = 1.$$

We claim that these $\alpha_1, \dots, \alpha_n$ satisfy (32). Let $(\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m$ be an arbitrary vector. Note that

$$\Phi(t) = \sum_{i=1}^m \lambda_i f_i(0) + \sum_{i=1}^m \sum_{j=1}^n \lambda_i a_{ij} e^{jt}.$$

Differentiating $k \geq 1$ times results

$$\Phi^{(k)}(t) = \sum_{i=1}^m \sum_{j=1}^n \lambda_i j^k a_{ij} e^{jt}.$$

Hence by (33) we obtain

$$\sum_{k=1}^n \alpha_k \Phi^{(k)}(t) = \sum_{k=1}^n \sum_{i=1}^m \sum_{j=1}^n \alpha_k j^k \lambda_i a_{ij} e^{jt} = \sum_{i=1}^m \sum_{j=1}^n \left(\sum_{k=1}^n \alpha_k j^k \right) \lambda_i a_{ij} e^{jt} = \Phi(t) - \sum_{i=1}^m \lambda_i f_i(0),$$

□

Set $J = \{a \leq t \leq T : |\Phi(t)| < 1/4\}$, let $\alpha_1, \dots, \alpha_n$ be provided by Lemma 5.5. Define $H := \max_{1 \leq i \leq n} |\alpha_i|$. We will show that $[a, T] \setminus J$ is a disjoint union of $O_n(1)$ intervals such that in each interval $\Phi'(t)$ is monotone and $|\Phi^{(k)}(t)| \geq 1/(8Hn)$ for a **fixed** $1 \leq k \leq n$.

We recall that $1, f_1, \dots, f_m$ are linearly independent and $(\lambda_1, \dots, \lambda_m)$ is not-zero. Thus $\sum_{i=1}^m \lambda_i f_i(t)$ is a polynomial of degree at least 1 and at most n . Hence from Lemma 5.3 we know that $[a, T] \setminus J$ can be written as a disjoint union of at most $O_n(1)$ intervals I'_1, \dots, I'_p such that $\Phi'(t)$ is monotone on each I'_i . Moreover notice that for each $1 \leq i \leq p$ we have

$$(34) \quad I'_i \subseteq \bigcup_{k=1}^n \left\{ t \in [a, T] : |\Phi^{(k)}(t)| \geq \frac{1}{8Hn} \right\}.$$

In fact, if for some $t \in [a, T] \setminus J$ we have $|\Phi^{(k)}(t)| < \frac{1}{8Hn}$ for all $1 \leq k \leq n$, then using (32) we obtain

$$|\Phi(t)| = \left| \sum_{i=1}^m \lambda_i f_i(0) + \sum_{k=1}^n \alpha_k \Phi^{(k)}(t) \right| < \left| \sum_{i=1}^m \lambda_i f_i(0) \right| + \frac{Hn}{8Hn} \leq \frac{1}{4},$$

which is a contradiction since $t \notin J$. For each $1 \leq k \leq n$ set

$$\mathcal{A}_k = \left\{ t \in [a, T] : |\Phi^{(k)}(t)| \geq 1/(8Hn) \right\}.$$

Again by Lemma 5.3, each \mathcal{A}_k is a union of $O_n(1)$ intervals and so by Lemma 5.2 there exists $O_n(1)$ disjoint intervals I''_1, \dots, I''_q such that

$$\bigcup_{k=1}^n \mathcal{A}_k = \bigcup_{i=1}^q I''_i,$$

and for each $1 \leq j \leq q$, there exists $1 \leq i \leq n$ such that $I''_j \subseteq \mathcal{A}_i$. Hence from (34) we deduce that

$$[a, T] \setminus J = \bigcup_{i,j} (I'_i \cap I''_j),$$

is a union of at most $O_n(1)$ disjoint intervals $I_{ij} := I'_i \cap I''_j$ such that in each interval $\Phi'(t)$ is monotone and $|\Phi^{(k)}(t)| \geq 1/(8Hn)$ for a fixed $1 \leq k \leq n$. Hence, by applying van der Corput lemma for each I_{ij} , there exists an absolute constant $C_1 > 0$, depending only on the polynomial $\{f_1, \dots, f_m\}$, such that

$$(35) \quad \left| \int_{I_{ij}} \cos(2\pi\Phi(t)) dt \right| \leq C_1.$$

Moreover since the intervals I_{ij} partition $[a, T] \setminus J$ into $O_n(1)$ intervals, then from (44) there exists another absolute constant $C_2 > 0$, depending only on the polynomial f ,

$$\left| \int_{[a, T] \setminus J} \cos(2\pi\Phi(t)) dt \right| \leq C_2.$$

Also it is immediate that

$$\int_J \cos(2\pi\Phi(t)) dt \geq 0,$$

since on J we have $|2\pi\Phi(t)| \leq \pi/2$. Therefore

$$-\frac{C_2}{T-a} \leq \widehat{\mu_T}(\lambda_1, \dots, \lambda_m),$$

when $|\sum_{i=1}^m \lambda_i f_i(0)| \leq 1/8$.

Remark 5.6. Pertaining to Remark 2.3, let $f_1(0) = \dots = f_m(0) = 0$, and assume that $(f_1(t), \dots, f_m(t))$ is non-zero for $t \neq 0$. Under this condition, we can indeed pick

$$a_0 = \sup\{t : f_1(e^t) = \dots = f_m(e^t) = 0\} = -\infty,$$

and carry out the proof. This in particular shows that if T is fixed and $a \rightarrow -\infty$, then

$$\chi_{\text{Bor}}(\text{Cay}(\mathbb{R}^m, \mathcal{S}_{a,T})) \rightarrow \infty.$$

5.2. High frequency case. Now we consider the case where

$$(36) \quad \left| \sum_{i=1}^m \lambda_i f_i(0) \right| > \frac{1}{8}.$$

As before, we let $n = \max_i \deg f_i$ and $f_i(t) = \sum_{j=0}^n a_{ij} t^j$. Since $1, f_1, \dots, f_m$ are linearly independent, at least one of the $m \times m$ submatrices of

$$A' = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

is invertible. Set

$$M := \max \left\{ \|A^{-1}\|_{\text{op}} : A \text{ is an invertible } m \times m \text{ submatrix of } A' \right\}, \quad L := \left(\sum_{i=1}^m f_i^2(0) \right)^{1/2}.$$

From (36) we have $L > 0$. We show that there exists $1 \leq \ell \leq n$ such that

$$(37) \quad \left| \sum_{i=1}^m \lambda_i a_{i\ell} \right| > \varepsilon := \frac{1}{8\sqrt{m}LM}.$$

Assume the contrary. Then $(\lambda_1, \dots, \lambda_m)A' \in [-\varepsilon, \varepsilon]^n$. Applying A^{-1} with A as in the above paragraph, we obtain

$$(38) \quad \|(\lambda_1, \dots, \lambda_m)\| \leq \|A^{-1}\|_{\text{op}} \sqrt{m}\varepsilon.$$

From the Cauchy–Schwarz inequality applied to (36) and (38) it follows that

$$\frac{1}{8L} < (\lambda_1^2 + \dots + \lambda_m)^{1/2} \leq M\sqrt{m}\varepsilon,$$

which is a contradiction. We will fix the value of ℓ throughout the rest of the argument.

Lemma 5.7. *There exist real numbers β_1, \dots, β_n , where $n = \max_i \deg f_i$, not all zero such that for every $(\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m$:*

$$(39) \quad \left(\sum_{i=1}^m \lambda_i a_{i\ell} \right) e^{\ell t} = \sum_{k=1}^n \beta_k \Phi^{(k)}(t)$$

Proof. Using the Lagrange interpolation method (or Vendermonde determinant) we find real numbers β_1, \dots, β_n , not all zero, such that

$$(40) \quad \sum_{k=1}^n \beta_k j^k = \begin{cases} 1 & j = \ell, \\ 0 & j \neq \ell. \end{cases}$$

Hence

$$\sum_{k=1}^n \beta_k \Phi^{(k)}(t) = \sum_{k=1}^n \sum_{i=1}^m \sum_{j=1}^n \beta_k j^k \lambda_i a_{ij} e^{jt} = \sum_{i=1}^m \sum_{j=1}^n \left(\sum_{k=1}^n \beta_k j^k \right) \lambda_i a_{ij} e^{jt} = \left(\sum_{i=1}^m \lambda_i a_{i\ell} \right) e^{\ell t}.$$

This finishes the proof of the lemma. \square

Set $H' = \max_{1 \leq k \leq n} |\beta_k|$, where β_1, \dots, β_n are provided by Lemma 5.7. We claim that for every $t \in [a, T]$, there exists $1 \leq k \leq n$ such that

$$(41) \quad |\Phi^{(k)}(t)| \geq \frac{\varepsilon}{nH'},$$

where ε is defined in (37). In fact, if this is not the case, we have

$$(42) \quad \left| \sum_{i=1}^m \lambda_i a_{i\ell} \right| \leq \left| \left(\sum_{i=1}^m \lambda_i a_{i\ell} \right) e^{\ell t} \right| = \left| \sum_{k=1}^n \beta_k \Phi^{(k)}(t) \right| \leq nH' \frac{\varepsilon}{nH'} = \varepsilon,$$

which is a contradiction since $|\sum_{i=1}^m \lambda_i a_{i\ell}| > \varepsilon$. For each $1 \leq k \leq n$, set

$$\mathcal{B}_k := \{t \in [a, T] : |\Phi^{(k)}(t)| \geq \varepsilon/nH'\}.$$

Since $\sum_{i=1}^m \lambda_i f_i(t)$ is a polynomial of degree at least 1 and at most n , it follows from Lemma 5.3 that \mathcal{B}_k can be written as a disjoint union of $O_n(1)$ intervals $I'_{k1}, \dots, I'_{kp_k}$ such that $\Phi'(t)$ is monotone on each I'_i . Applying Lemma 5.2 to $\{I'_{kj} : 1 \leq k \leq n, 1 \leq j \leq p_k\}$ we find $O_n(1)$ disjoint intervals I''_1, \dots, I''_q such that

$$(43) \quad \bigcup_{k=1}^n \mathcal{B}_k = \bigcup_{i=1}^q I''_i,$$

and in each interval I''_i the function $\Phi'(t)$ is monotone and $|\Phi^{(k)}(t)| \geq \varepsilon/(H'n)$ for some fixed $1 \leq k \leq n$ depending on i . Hence, by applying van der Corput lemma for each I''_i , we can find a constant $C_3 > 0$, depending only on the polynomial $\{f_1, \dots, f_m\}$, such that

$$(44) \quad \left| \int_{I''_i} \cos(2\pi\Phi(t)) dt \right| \leq C_3.$$

Arguing as in the previous case, we can find an absolute constant $C_4 > 0$, depending only on the polynomials f_1, \dots, f_m , such that

$$-\frac{C_4}{T-a} \leq \widehat{\mu}_T(\lambda_1, \dots, \lambda_m),$$

when $|\sum_{i=1}^m \lambda_i f_i(0)| > 1/8$. This finishes the proof of the second case and thus the proof of Theorem 5.1. \square

By Remark 4.4 and Theorem 5.1 we deduce that

$$-\frac{C}{T-a} \leq \inf_{u \in \mathbb{R}^m} \widehat{\mu}_T(u) < 0,$$

for some absolute constant $C > 0$. This along with Theorem 4.8 proves Theorem 2.1 for $\mathbb{K} = \mathbb{R}$.

6. OSCILLATORY INTEGRALS WITH p -ADIC POLYNOMIAL PHASE

We now turn to the proof of Theorem 2.1 in the p -adic case. Denote the ring of integers of \mathbb{Q}_p by \mathbb{Z}_p and set $\mathfrak{p} = p\mathbb{Z}_p$. From here we have

$$\mathfrak{p}^k = \{s \in \mathbb{Q}_p : \|s\|_p \leq p^{-k}\},$$

and we have the filtration

$$\mathbb{Q}_p \supseteq \dots \supseteq \mathfrak{p}^{-2} \supseteq \mathfrak{p}^{-1} \supseteq \mathfrak{p}^0 = \mathbb{Z}_p \supseteq \mathfrak{p}^1 \supseteq \mathfrak{p}^2 \supseteq \dots$$

For $k \in \mathbb{Z}$, denote the sphere of radius p^k by

$$\mathcal{C}_k := \{s \in \mathbb{Q}_p : \|s\|_p = p^k\}.$$

Notice that $\mathcal{C}_k = \mathfrak{p}^{-k} \setminus \mathfrak{p}^{-(k-1)}$, and so $|\mathcal{C}_k| = p^k - p^{k-1}$, where $|\mathcal{C}_k|$ denotes the Haar measure of \mathcal{C}_k .

Definition 6.1. Let $f(x) = a_n x^n + \dots + a_1 x + a_0 \in \mathbb{Q}_p[x]$ be a polynomial with $a_n \neq 0$ and $n \geq 1$. The essential part of f is defined by

$$(45) \quad \text{Ess}_f := \max \left\{ 0, \log_p \left(\frac{\|a_i\|_p}{\|a_n\|_p} \right) : 0 \neq a_i, 0 \leq i \leq n-1 \right\}.$$

From the ultrametric property of the p -adic norm it follows that if $\|s\|_p \geq p^a$ and $a > \text{Ess}_f$ then

$$(46) \quad \|f(s)\|_p = \|a_n s^n\|_p.$$

Let ψ be the Tate character defined by (20).

Lemma 6.2. Let $f(x) \in \mathbb{Q}_p[x]$ be a polynomial with $\deg f \geq 1$. Then for any $\lambda \in \mathbb{Q}_p$ and any integer T with $T > a > \text{Ess}_f$ we have

$$(47) \quad -16p^{\deg f} \leq \int_{p^a \leq \|s\|_p \leq p^T} \frac{\psi(\lambda f(s)) + \bar{\psi}(\lambda f(s))}{\|s\|_p} ds.$$

Similar to the real case, this p -adic oscillatory integral can be estimated by a suitable van der Corput lemma for p -adic integrals.

Theorem 6.3 (p -adic van der Corput's Lemma). Suppose that $a_0, \dots, a_n \in \mathbb{Q}_p$, where $a_n \neq 0$ and $n \geq 1$. Then for all $r \in \mathbb{Z}$, we have

$$\left| \int_{\mathfrak{p}^r} \psi(a_0 + a_1 s + \dots + a_n s^n) ds \right| \leq \frac{2p^n}{\|a_n\|_p^{1/n}}$$

Proof. See [24, Corollary 5]. \square

Now by exploiting the p -adic van der Corput lemma we prove the above theorem.

Proof of Lemma 6.2. Let $f(x) = a_n x^n + \dots + a_1 x + a_0$ be the given polynomial. The inequality is evident if $\lambda = 0$. Hence assume that $\lambda \neq 0$ and let $\|\lambda a_n\|_p = p^\ell$ where $\ell \in \mathbb{Z}$. We claim that $\psi(\lambda f(s)) = 1$ whenever $p^a \leq \|s\|_p \leq p^{k_0}$ where $k_0 := \lfloor -\frac{\ell}{n} \rfloor$. To see this, notice that $a > \text{Ess}_f$ and for any $s \in \mathcal{C}_k$ with $a \leq k \leq k_0$ we have

$$(48) \quad \|\lambda f(s)\|_p = \|\lambda a_n s^n\|_p = p^{\ell + nk} \leq 1,$$

which proves the claim since $\mathbb{Z}_p = \ker \psi$. Set $k_1 = \max\{k_0 + 1, a\}$. Therefore

$$(49) \quad \int_{p^a \leq \|s\|_p \leq p^T} \frac{\psi(\lambda f(s)) + \bar{\psi}(\lambda f(s))}{\|s\|_p} ds \geq \int_{p^{k_1} \leq \|s\|_p \leq p^T} \frac{\psi(\lambda f(s)) + \bar{\psi}(\lambda f(s))}{\|s\|_p} ds.$$

It thus suffices to find a lower bound for the right hand side of the above inequality. Recall that $\mathcal{C}_r = \{s \in \mathbb{Q}_p : \|s\|_p = p^r\}$. We have

$$(50) \quad \begin{aligned} \left| \int_{p^{k_1} \leq \|s\|_p \leq p^T} \frac{\psi(\lambda f(s)) + \bar{\psi}(\lambda f(s))}{\|s\|_p} ds \right| &\leq \sum_{r=k_1}^T \left| \int_{\mathcal{C}_r} \frac{\psi(\lambda f(s)) + \bar{\psi}(\lambda f(s))}{\|s\|_p} ds \right| \\ &= \sum_{r=k_1}^T \frac{1}{p^r} \left| \int_{\mathcal{C}_r} (\psi(\lambda f(s)) + \bar{\psi}(\lambda f(s))) ds \right| \\ &\leq 2 \sum_{r=k_1}^T \frac{1}{p^r} \left| \int_{\mathcal{C}_r} \psi(\lambda f(s)) ds \right|. \end{aligned}$$

Notice that $\mathcal{C}_r = \mathfrak{p}^{-r} \setminus \mathfrak{p}^{-(r-1)}$. Hence by the p -adic van der Corput lemma we have

$$(51) \quad \left| \int_{\mathcal{C}_r} \psi(\lambda f(s)) ds \right| = \left| \int_{\mathfrak{p}^{-r}} \psi(\lambda f(s)) ds - \int_{\mathfrak{p}^{-(r-1)}} \psi(\lambda f(s)) ds \right| \leq \frac{4p^n}{\|\lambda a_n\|_p^{1/n}}.$$

From (50) and (51) we conclude that

$$\left| \int_{p^{k_1} \leq \|s\|_p \leq p^T} \frac{\psi(\lambda f(s)) + \bar{\psi}(\lambda f(s))}{\|s\|_p} ds \right| \leq \frac{8p^n}{p^{\ell/n}} \sum_{r=p^{k_1}}^T \frac{1}{p^r} \leq \frac{8p^n}{p^{\ell/n+k_1}} \sum_{r \geq 0} \frac{1}{p^r} \leq 16p^n,$$

since $\ell/n + k_1 \geq 0$. This inequality along with (49) provide the lower bound. \square

Now we return to the proof of Theorem 2.1 for p -adic case. Similar to the real case, let $f_1(x), \dots, f_m(x)$ be a family of $m \geq 2$ polynomials in $\mathbb{Q}_p[x]$. Define the following symmetric, bounded Borel set

$$(52) \quad \mathcal{S}_T := \mathcal{S}_{a,T,f_1,\dots,f_m} = \{\pm(f_1(s), \dots, f_m(s)) \in \mathbb{Q}_p^m : p^a \leq \|s\|_p \leq p^T\}.$$

Let $n = \max_{1 \leq i \leq m} \deg f_i$ and $f_i(x) = \sum_{j=0}^n a_{ij}x^j$. Since, by our assumption, $1, f_1, \dots, f_m$ are linearly independent, the rank of the coordinate matrix of $f_1, \dots, f_m, 1$ with respect to the ordered basis $\{x^n, \dots, x, 1\}$ given by

$$A = \begin{pmatrix} a_{1,n} & a_{1,n-1} & \cdots & a_{1,0} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,n} & a_{m,n-1} & \cdots & a_{m,0} \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

is $m+1$. Thus by applying the Gauss elimination method, A can be reduced to a matrix in the row echelon form of rank $m+1$ with the same last row as A . Hence there exists $B \in \text{GL}_m(\mathbb{Q}_p)$ such that

$$B \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} = \begin{pmatrix} f'_1 \\ \vdots \\ f'_m \end{pmatrix},$$

with $\deg f'_1 > \deg f'_2 > \cdots > \deg f'_m \geq 1$. Since $B \in \text{GL}_m(\mathbb{Q}_p)$, then one can find an absolute constant $c > 0$, depending only on f_1, \dots, f_m such that

$$c\bar{\alpha}(\text{Cay}(\mathbb{Q}_p^m, \mathcal{S}_T)) \leq \bar{\alpha}(\text{Cay}(\mathbb{Q}_p^m, B(\mathcal{S}_T))) = \bar{\alpha}(\text{Cay}(\mathbb{Q}_p^m, \mathcal{S}_{a,T,f'_1,\dots,f'_m})).$$

So we may and will assume from now on that $\deg f_1 > \deg f_2 > \cdots > \deg f_m \geq 1$. We will also assume that a is a positive integer with

$$a > a_0 := \max\{\text{Ess}_{f_i} : 1 \leq i \leq m\},$$

in (52) and so $0 \notin \overline{\mathcal{S}_T}$. With these assumptions, for a measurable set $E \subseteq \mathbb{Q}_p^m$ we define the following probability measure

$$\mu_T(E) = \frac{1}{L} \int_{p^a \leq \|s\|_p \leq p^T} \frac{\mathbf{1}_E(f_1(s), \dots, f_m(s)) + \mathbf{1}_{(-E)}(f_1(s), \dots, f_m(s))}{\|s\|_p} ds,$$

where

$$L = 2(T - a + 1) \left(1 - \frac{1}{p}\right).$$

It is easy to verify that μ_T is a symmetric measure with $\text{Supp}(\mu_T) = \mathcal{S}_T$. By a straightforward calculation we obtain

$$\widehat{\mu}_T(\lambda_1, \dots, \lambda_m) = \frac{1}{L} \int_{p^a \leq \|s\|_p \leq p^T} (\psi(\lambda_1 f_1(s) + \dots + \lambda_m f_m(s)) + \bar{\psi}(\lambda_1 f_1(s) + \dots + \lambda_m f_m(s))) \frac{ds}{\|s\|_p},$$

where $(\lambda_1, \dots, \lambda_m) \in \mathbb{Q}_p^m$. Our aim now is to estimate $\widehat{\mu}_T(\lambda_1, \dots, \lambda_m)$.

Theorem 6.4. *For any integer $T > a > a_0$ and arbitrary $(\lambda_1, \dots, \lambda_m) \in \mathbb{Q}_p^m$ we have*

$$(53) \quad -\frac{16 \sum_{i=1}^m p^{\deg f_i}}{L} \leq \widehat{\mu}_T(\lambda_1, \dots, \lambda_m).$$

Proof of Theorem 6.4. Lemma 6.2 verifies (53) when $m = 1$. We prove the theorem by induction on m . Let $f_i(x) = \sum_{j=0}^{n_i} a_{ij} x^j$ and set

$$\Psi_{f_1, \dots, f_m}(s) := \psi(\lambda_1 f_1(s) + \dots + \lambda_m f_m(s)) + \bar{\psi}(\lambda_1 f_1(s) + \dots + \lambda_m f_m(s)).$$

By the induction hypothesis we can assume that $\lambda_1 \neq 0$. Now similar to Lemma 6.2, assume that $\|\lambda_1 a_{1n_1}\| = p^\ell$ and then we have $\psi(\lambda_1 f_1(s)) = 1$ whenever $p^a \leq \|s\|_p \leq p^{k_0}$, where $k \leq k_0 := \lfloor -\frac{\ell}{n_1} \rfloor$. Set $k_1 = \max\{k_0 + 1, a\}$. Thus from the induction hypothesis we obtain

$$(54) \quad \begin{aligned} \int_{p^a \leq \|s\|_p \leq p^T} \frac{\Psi_{f_1, \dots, f_m}(s)}{\|s\|_p} ds &= \int_{p^a \leq \|s\|_p \leq p^{k_1-1}} \frac{\Psi_{f_2, \dots, f_m}(s)}{\|s\|_p} ds + \int_{p^{k_1} \leq \|s\|_p \leq p^T} \frac{\Psi_{f_1, \dots, f_m}(s)}{\|s\|_p} ds \\ &\geq -16 \sum_{2 \leq i \leq m} p^{\deg f_i} + \int_{p^{k_1} \leq \|s\|_p \leq p^T} \frac{\Psi_{f_1, \dots, f_m}(s)}{\|s\|_p} ds. \end{aligned}$$

Note that since $\deg f_1 > \dots > \deg f_m \geq 1$, as in Lemma 6.2, we apply the p -adic van der Corput lemma to obtain

$$(55) \quad \begin{aligned} \left| \int_{p^{k_1} \leq \|s\|_p \leq p^T} \frac{\Psi_{f_1, \dots, f_m}(s)}{\|s\|_p} ds \right| &\leq \sum_{r=k_1}^T \frac{1}{p^r} \left| \int_{C_r} \Psi_{f_1, \dots, f_m}(s) ds \right| \\ &\leq \frac{8p^{n_1}}{\|\lambda_1 a_{1n_1}\|_p^{1/n_1}} \sum_{r=k_1}^{\infty} \frac{1}{p^r} \leq 16p^{n_1}. \end{aligned}$$

This along with (54) proves Theorem 6.4. □

Now by Remark 4.4 and Theorem 6.4 we deduce that

$$-\frac{C}{T-a} \leq \inf_{u \in \mathbb{Q}_p^m} \widehat{\mu}_T(u) < 0,$$

for some absolute constant $C > 0$. This along with Theorem 4.8 proves Theorem 2.1 for $\mathbb{K} = \mathbb{Q}_p$.

7. AFFINE BÉZOUT THEOREM AND PROOF OF THEOREM 2.11

In this section, we will prove Theorem 2.11. Before starting the proof, we will recall the following affine version of Bézout's theorem. We assume that K is an algebraic closed field of characteristic zero.

Theorem 7.1. *Let $V_1, V_2 \subseteq K^m$ be two one-dimensional irreducible varieties defined by polynomials P_1, \dots, P_r with $\deg P_i = d_i$ (respectively P'_1, \dots, P'_s with $\deg P' = d'_i$). Then either $V_1 = V_2$ or*

$$|V_1 \cap V_2| \leq d_1 \dots d_r d'_1 \dots d'_s.$$

Proof. Notice that $V_1 \cap V_2$ is a finite set and $r + s \geq m$. Hence by [26, Theorem 3.1] we obtain the inequality. For more details see also [31, Theorem 5]. □

Recall that the Ramsey number $R(d, d)$ is the least integer m such that for any edge coloring of the complete graph K_m in red and blue, there exist d vertices forming a monochromatic K_d . Let $V \subseteq \mathbb{A}_K^m$ be an irreducible variety of dimension 1 that is not an affine line. Moreover assume that V is defined by $P_1 = \dots = P_n = 0$, where $P_i \in K[x_1, \dots, x_m]$ with $d_i = \deg P_i$. Set $d = (d_1 \dots d_n)^2 + 3$ and $m = R(d, d)$. Suppose $\mathcal{G} = \text{Cay}(K^m, \pm V)$ contains a copy of the complete graph K_m formed with vertices u_1, \dots, u_m . For $1 \leq i < j \leq m$, color the edge between u_i and u_j red if $u_j - u_i \in V$, and blue if $u_j - u_i \in (-V) \setminus V$. In view of $m = R(d, d)$, there exists a monochromatic complete graph on d vertices. Without loss of generality, we will assume that it is a red K_d with vertices u_1, \dots, u_d , thus $u_j - u_i \in V$ for $j > i$. Consider the variety $V' = V + (u_2 - u_1)$. Note that for $3 \leq i \leq d$, we have $u_i - u_1 \in V$ and $u_i - u_1 = (u_i - u_2) + (u_2 - u_1) \in V + (u_2 - u_1) = V'$. This implies that V and V' have at least $d - 2 > (d_1 \dots d_n)^2$ intersection points. Hence by Theorem 7.1 we have $V = V + (u_2 - u_1)$. If v is an arbitrary point on V , it follows that V contains the points $v - n(u_2 - u_1)$ for all integers $n \geq 1$. Hence V contains the Zariski closure of these points, which is a line. Since V is irreducible, it follows that V is a line, which is a contrary to the hypotheses. Therefore $\omega(\text{Cay}(K^m, \pm V)) < m$.

8. SOME REMARKS ON CAYLEY GRAPHS OF CURVES

In this section, we will investigate the question of coloring for Cayley graphs with respect to curves other than the ones studied above. For simplicity, we will restrict the discussion to the case $n = 2$. Extension to the general case is straightforward. Let $F : \mathbb{R} \rightarrow \mathbb{R}^2$ be a continuous function. Set

$$\Gamma_F := \text{Cay}(\mathbb{R}^2, \mathcal{S}_F), \quad \mathcal{S}_F = \{\pm F(s) : s \in \mathbb{R}\}.$$

Remark 8.1. For $b > a > 0$, and a linear functional $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$, let $H_{a,b}$ denote the region defined by $a < \phi(x) < b$. Hence, if $\phi(F(t)) \in (a, b)$, then

$$\chi_{\text{Bor}}(\Gamma_F) \leq \chi_{\text{Bor}}(\text{Cay}(\mathbb{R}^2, \pm H_{a,b})) \leq \chi_{\text{Bor}}(\text{Cay}(\mathbb{R}, \pm(a, b))) < \infty.$$

This property is clearly satisfied by some algebraic curves. Presumably, the condition of being unbounded in every direction could be enough for algebraic sets to satisfy (3).

Proposition 8.2. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous periodic function with $f(0) \neq 0$. If $F(t) = (t, f(t))$, then $\chi_{\text{Bor}}(\Gamma_F) < \infty$.

Proof. Without loss of generality we can assume that the period of the function is 1. Since $f(0) \neq 0$ then for some $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x| \leq \delta$ then $\varepsilon < |f(x)|$. Assume $\|f\|_\infty \leq M$ and pick an integer $n > \max(M + 2, \varepsilon^{-1}, \delta^{-1})$. Define the following Borel measurable function

$$c : \mathbb{R}^2 \rightarrow \mathbb{Z}/(n\mathbb{Z}) \times \mathbb{Z}/n^2\mathbb{Z}, \quad (x, y) \mapsto ([nx] \pmod{n}, [ny] \pmod{n^2}).$$

Suppose $c(x, y) = c(x', y')$. Therefore $[ny'] = n^2k + [ny]$ for some $k \in \mathbb{Z}^{\geq 0}$. First assume $k > 0$. Then $|y' - y| > M$ and so (x, y) and (x', y') do not form an edge. Hence assume $k = 0$ and so

$$(56) \quad |y' - y| < 1/n < \varepsilon$$

Moreover we have $[nx'] = nk_1 + [nx]$ for some $k_1 \in \mathbb{Z}$. Let $nx = [nx] + \theta$ and $nx' = [nx'] + \theta'$, where $\theta, \theta' \in [0, 1)$. This implies that $|x - x' - k_1| \leq 1/n < \delta$. By periodicity, $|f(x' - x)| > \varepsilon$. From here and (56) we deduce that (x, y) and (x', y') can not be an edge. \square

Note that the graph of a continuous periodic function is also bounded by two hyperplanes. In the rest of section, we will prove Theorem 2.12

Proof of Theorem 2.12. Let $\Lambda = \{q_n = 3^{n-1} : n \geq 1\}$. Then $\chi(\text{Cay}(\mathbb{Z}, \pm \Lambda)) = 2$ with the color classes $2\mathbb{Z}$ and $2\mathbb{Z} + 1$. Let $c : \mathbb{Z} \rightarrow \{1, 2\}$ be the given coloring map. Let $I = [-1/2, 1/2)$, and for $x \in I$, define $\omega(x) = -1$ for $x < 0$ and $\omega(x) = 1$ for $x \geq 0$. For every $x \in \mathbb{R}$, write $x = z(x) + h(x)$ with $z(x) \in \mathbb{Z}$ and $h(x) \in I$. Consider the map

$$\tilde{c}(x) = (c(z(x)), \omega(h(x))).$$

We claim that if $\tilde{c}(x_1) = \tilde{c}(x_2)$ then for all $n \geq 1$ we have $|x_1 - x_2 - q_n| > 1/4$. Assume the contrary, that is, $\tilde{c}(x_1) = \tilde{c}(x_2)$ and that $|x_1 - x_2 - q_n| \leq 1/4$. It follows that

$$(57) \quad \begin{aligned} |z(x_1) - z(x_2) - q_n| &= |(x_1 - x_2 - q_n) - (h(x_1) - h(x_2))| \\ &\leq |x_1 - x_2 - q_n| + |h(x_1) - h(x_2)| \leq 1/4 + 1/2 < 1. \end{aligned}$$

This implies that $z(x_1) - z(x_2) = q_n$, which is a contradiction to $c(z(x_1)) = c(z(x_2))$. Now, consider the region S defined by

$$S = \{(x, \pm q_n + y) : x \in \mathbb{R}, |y| < 1/4\} \cup \{(\pm q_n + x, y) : y \in \mathbb{R}, |x| < 1/4\}.$$

We claim that $\text{Cay}(\mathbb{R}^2, S)$ has a finite Borel chromatic number. In fact, for each $(x, y) \in \mathbb{R}^2$, set $c_2(x, y) = (\tilde{c}(x), \tilde{c}(y))$. Suppose $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ are such that $(x_1 - x_2, y_1 - y_2) \in S$. Then there exists $n \geq 1$ such that $|x_1 - x_2 - q_n| < 1/4$ or $|y_1 - y_2 - q_n| < 1/4$. In the former case, we have $\tilde{c}(x_1) \neq \tilde{c}(x_2)$ and in the latter case, we have $\tilde{c}(y_1) \neq \tilde{c}(y_2)$. From this we conclude that $c_2(x_1, y_1) \neq c_2(x_2, y_2)$.

It now remains to see that S contains the graph of an analytic function which is not bounded between any two parallel lines. Define the piecewise linear echelon-shaped curve L by

$$L = \bigcup_{n=1}^{\infty} \{(x, q_n) : q_n \leq x \leq q_{n+1}\} \cup \bigcup_{n=1}^{\infty} \{(q_{n+1}, y) : q_n \leq y \leq q_{n+1}\}.$$

Note that L does not lie in the region between any two parallel lines. Identify \mathbb{R}^2 with the complex plane. Let S_1 be the δ -neighborhood of L for $\delta < 1/4$. It is easy to see that $S_1 \subset S$ and S_1 is a simply connected domain whose boundary viewed as a subset of the Riemann sphere is a Jordan curve containing ∞ . Let \mathbb{D} denote the unit disk in \mathbb{C} . By Caratheodory's extension of Riemann mapping theorem, which says any conformal mapping between two Jordan regions can be extended to a homeomorphism between the closures of the two regions, the uniformizing map $h : \mathbb{D} \rightarrow S_1$ extends to a homeomorphism $\tilde{h} : \overline{\mathbb{D}} \rightarrow \overline{S_1}$, mapping the unit circle $\partial\mathbb{D}$ onto ∂S_1 . Thus there is a point in $\partial\mathbb{D}$ which maps to ∞ . Without loss of generality, we can assume that $h(1) = \infty$. Then the analytic curve $h : (0, 1) \rightarrow S_1$ provides the required real analytic curve. \square

Remark 8.3. The above construction has the following generalization. Let $\Lambda = \{q_n\}_{n \geq 1}$ be an increasing lacunary sequence, that is, $q_{n+1}/q_n \geq 1 + \varepsilon$ for some $\varepsilon > 0$. In [18], Katznelson proved that under these hypotheses we have $\chi(\text{Cay}(\mathbb{Z}, \pm\Lambda)) < \infty$. By applying this theorem, the set $\{3^{n-1} : n \geq 1\}$ in the above proof, can be replaced by any lacunary sequence Λ .

It would be desirable to obtain an adequate description of the sets $\mathcal{S} \subseteq \mathbb{R}^m$ of Hausdorff dimension at least one for which the Borel chromatic number of $\text{Cay}(\mathbb{R}^m, \mathcal{S})$ is infinite. In particular, it would be interesting to know the answer to the following two questions:

Question 2. Suppose \mathcal{S} is an irreducible Zariski closed subset of \mathbb{C}^m which is not contained in an affine hyperplane. Is it true that $\chi_{\text{Bor}}(\mathbb{C}^m, \pm\mathcal{S})$ is infinite? Note that \mathcal{S} is automatically non-compact.

Question 3. Suppose \mathcal{S}_0 is an irreducible Zariski closed subset of \mathbb{R}^m which is not contained in an affine hyperplane. Let $d_i \rightarrow \infty$ and $\mathcal{S} = \bigcup_{n=1}^{\infty} d_n \mathcal{S}_0$ be a union of dilations of \mathcal{S}_0 . Is it true that $\chi_{\text{Bor}}(\mathbb{R}^m, \pm\mathcal{S})$ is infinite?

9. ALGEBRAIC ASPECTS OF CAYLEY GRAPHS

This short section is devoted to the proof of Theorem 2.13. The following theorem due to Cassels [10] will be needed in the proof.

Theorem 9.1. *Let K be a finitely generated extension of \mathbb{Q} and $C \subseteq K \setminus \{0\}$ be a finite set. Then there exist infinitely many primes p for which there exists an embedding $\iota : K \rightarrow \mathbb{Q}_p$ such that $\iota(x) \in \mathbb{Z}_p \setminus p\mathbb{Z}_p$ for all $x \in C$.*

Moreover we recall the following theorem of de Bruijn and Erdős [12, Theorem 8.1.3].

Theorem 9.2. *Let $G = (V, E)$ be a graph and $k \in \mathbb{N}$. If every finite subgraph of G has chromatic number at most k , then so does G .*

We now are ready to prove Theorem 2.13.

Proof of Theorem 2.13. Recall that V defined by $F_1 = \cdots = F_n = 0$ is an algebraic set defined over \mathbb{Z} . For every prime p , the algebraic closure of \mathbb{Q}_p is an algebraic closed field of characteristic zero and transcendental degree 2^{\aleph_0} over \mathbb{Q} , and hence, by a well-known theorem of Steinitz, is isomorphic to \mathbb{C} . This yields a ring embedding $i : \mathbb{Q}_p \hookrightarrow \mathbb{C}$, implying that

$$\sup_p \chi(\text{Cay}(\mathbb{Z}_p^m, \pm V(\mathbb{Z}_p))) \leq \chi(\text{Cay}(\mathbb{C}^m, \pm V(\mathbb{C}))).$$

Note that the embedding i is purely algebraic and is far from being continuous or even measurable.

To prove the reverse inequality, by applying Theorem 9.2, it suffices to show that for any finite subgraph \mathcal{G} of $\text{Cay}(\mathbb{C}^m, \pm V(\mathbb{C}))$, there exists a prime p such that $\chi(\mathcal{G}) \leq \chi(\text{Cay}(\mathbb{Z}_p^m, \pm V(\mathbb{Z}_p)))$. Assume that $v \in \mathbb{C}^m$ are the vertices of \mathcal{G} . For any edge $vw \in E(\mathcal{G})$, we have

$$F_j(v - w) = 0 \quad \text{or} \quad F_j(w - v) = 0.$$

Also, as v are pairwise distinct vectors, for any two vertices v, w there exists a vector $\mathbf{z}_{v,w} \in \mathbb{C}^m$ such that

$$(v - w) \cdot \mathbf{z}_{v,w} = 1,$$

where \cdot denotes the inner product. Denote by C the set of all entries of v for all $v \in V(\mathcal{G})$ and $\mathbf{z}_{v,w}$ for all pairs of vertices v, w . Let K be the finitely generated extension of \mathbb{Q} generated by C . Note that, viewed as vertices of $\text{Cay}(K^m, \pm V(K))$, the points v span a subgraph isomorphic to \mathcal{G} . We now apply Theorem 9.1 to find a prime p and a field embedding $\iota : K \hookrightarrow \mathbb{Q}_p$, such that $\iota(C) \subseteq \mathbb{Z}_p$. As ι is a field embedding, we have

$$F_j(\iota(v) - \iota(w)) = 0 \quad \text{or} \quad F_j(\iota(w) - \iota(v)) = 0,$$

for all $1 \leq j \leq n$ and $vw \in E(\mathcal{G})$. Moreover since $(\iota(v) - \iota(w)) \cdot \iota(\mathbf{z}_{v,w}) = 1$, it follows that $\iota(v) \neq \iota(w)$. Thus \mathcal{G} embeds in $\text{Cay}(\mathbb{Z}_p^m, \pm V(\mathbb{Z}_p))$ which finishes the proof of the theorem. \square

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